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Boolean Algebra

Let $\{B, +, *, ', 0, 1\}$ be an algebraic structure with operations namely addition, multiplication, Unary (complement) and 0 and 1 are called zero element and unit element in B, then B is called Boolean algebra if for every $a, b, c \in B$, following conditions are satisfied:

(1) Commutative law.

$a+b = b+a$, $a*b = b*a$, $\forall a, b \in B$

(2) Identity law.

$a+0 = a$ and $a*1 = a$, $\forall a \in B$

(3) Distributive law.

$a+(b*c) = (a+b)*(a+c)$, $\forall a, b, c \in B$

$a*(b+c) = (a*b)+(a*c)$, $\forall a, b, c \in B$

(4) Complement law. Let $a \in B$ then a' is called complement of a if

$a+a' = 1$ and $a*a' = 0$, $\forall a \in B$

a' is called complement of a.

Remember. (1) $+$ or $\vee$ or $\cup$ or lcm
$*$ or $\wedge$ or $\cap$ or gcd
are parallel symbols

(2) zero element is least element
Unit element is greatest element
satisfying $a+a' =$ greatest element
$a*a' =$ least element

(3) $a' = \dfrac{\text{greatest element}}{a}$ in case of divisibility

and $a+b =$ l.c.m of a & b , $a*b =$ g.c.d of a & b.

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Examples of Boolean algebra

① Let S be a non-empty set then
$\{P(S) = \text{Power set of } S, \cup, \cap, ', \phi, S\}$
is a Boolean algebra
Here Union and intersection as two binary operations "+" and "×" respectively and complement of a set with respect to S as Unary operation $'$ (called complement), ϕ and S will work as 0 and 1 respectively i.e zero element and unity element.
Let $A \subseteq S$ then $A' = S - A$
To show it follows the following laws [C.I.D.C.]

(1) Commutativity: We know if $A, B \in P(S)$
then $A \cup B = B \cup A$ and $A \cap B = B \cap A$
$, \forall A, B \in P(S)$.

(2) Identity law. We know if zero element ϕ and unit element $1 \in P(S)$ then
$A \cup \phi = A$ and $A \cap S = A, \forall A \in P(S)$.

(3) Distributivety. Let $A, B, C \in P(S)$ then clearly
$(A \cup B) \cup C = A \cup (B \cup C), \forall A, B, C \in P(S)$
and $(A \cap B) \cap C = A \cap (B \cap C), \forall A, B, C \in P(S)$

(4) Complement law. For any $A \in P(S)$
$A' = (S - A) \in P(S)$ such that
$A \cup A' = S$ and $A \cap A' = \phi$

Hence $\{P(S), \cup, \cap, ', \phi, S\}$ is a Boolian algebra.

(2) Let $B = \{1, 2, 5, 7, 10, 14, 35, 70\}$. For any $a, b \in B$ we define "+" and "*" and unary operation ′ as follows

$a + b =$ l.c.m. of a and b
$a * b =$ g.c.d. of a and b

and $a' = \frac{70}{a}$.

Show $\{B, +, *, ', 1, 70\}$ is a Boolean algebra with 1 as zero element and 70 as unit element.

Solution. Let $B = \{1, 2, 5, 7, 10, 14, 35, 70\}$

"+" is defined as
$a + b =$ l.c.m of a and b

"*" is defined as
$a * b =$ g.c.d of a and b

Unary operation "′" is defined as
$a' = \frac{70}{a}$ [i.e $\frac{\text{unit element}}{a} = a'$
$a' =$ complement of a]

To show $\{B, +, *, ', 1, 70\}$ is a Boolean algebra.

Composition table for "+"

'+'	1	2	5	7	10	14	35	70
1	1	2	5	7	10	14	35	70
2	2	2	10	14	10	14	70	70
5	5	10	5	35	10	70	35	70
7	7	14	35	7	70	14	35	70
10	10	10	10	70	10	70	70	70
14	14	14	70	14	70	14	70	70
35	35	70	35	35	70	70	35	70
70	70	70	70	70	70	70	70	70

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Composition table for "*"

*	1	2	5	7	10	14	35	70
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
5	1	1	5	1	5	1	5	5
7	1	1	1	7	1	7	7	7
10	1	2	5	1	10	2	5	10
14	1	2	1	7	2	14	7	14
35	1	1	5	7	5	7	35	35
70	1	2	5	7	10	14	35	70

Composition table for " ' "

'	1	2	5	7	10	14	35	70
$a' = \frac{70}{a}$	70	35	14	10	7	5	2	1

Here we observe from C.T that following laws holds [C.I.D.C]

(1) Commutative laws

In C.T for "+" and in C.T for "*" } R_1 equals C_1
R_2 equals C_2
R_3 equals C_3
R_4 equals C_4
R_5 equals C_5
R_6 equals C_6
R_7 equals C_7
R_8 equals C_8

i.e $a+b=b+a$, $\forall\ a,b \in B$
and $a*b=b*a$, $\forall\ a,b \in B$
$\Rightarrow$ "+" and "*" are commutative in B.

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(2) Identity law. We observe

(i) in C.T for "+"

First row headed by 1 equals to top row

$\Rightarrow$ "1" is identity element in B

i.e $1 + a = a$, $\forall a \in B$

(ii) In C.T for "*"

In first row:

$1 * a = a$, $\forall a \in B$

$\Rightarrow$ 1 is identity element for "*" also.

(3) Distributive law. For any $a, b, c \in B$, we observe that

(i) $a + (b * c) = (a+b) * (a+c)$, $\forall a, b, c \in B$

and (ii) $a * (b+c) = (a*b) + (a*c)$, $\forall a, b, c \in B$

let $a = 2$, $b = 5$, $c = 7$

$a + (b * c) = 2 + (5 * 7) = 2 + 1 = 2$ [g.c.d of 5 & 7 = 1]

$(a+b) * (a+c) = (2+5) * (2+7)$

$= \underset{\text{l.c.m of } 2 \& 5}{10} * \underset{\text{l.c.m of } 2 \& 7}{14}$ = g.c.d of 10 & 14

$= 2$

$\therefore a + (b * c) = (a+b) * (a+c)$

Similarly we can verify for other values of a, b, c.

(4) Complement law

Here $a + a' = 70$ and $a * a' = 1$

where $a' = \frac{70}{a}$

$\therefore \{B, +, *, ', 1, 70\}$ is a Boolean algebra.

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Note. In Boolean algebra B following properties hold good: (I, B, A, A.)

① Idempotent law   Let $a \in B$.

(a) $a + a = a$ (b) $a * a = a$

② **Boundedness law**   Let $a \in B$

(a) $a + 1 = 1$, 1 is unit element (greatest element)

(b) $a * 0 = 0$, 0 is zero element (least element)

③ **Absorption law**   Let $a, b \in B$

(a) $a * (a+b) = a$

(b) $a + (a*b) = a$

④ **Associative law**   Let $a, b, c \in B$

$(a+b)+c = a+(b+c)$

$(a*b)*c = a*(b*c)$

Theorem. Let $\{B, +, *, ', 0, 1\}$ be a Boolean algebra then prove the following

① $a+a=a$ ② $a*a=a$ [Idempotent law]

(3) $a+1=1$ (4) $a*0=0$ [boundedness law]

⑤ $a*(a+b)=a$ ⑥ $a+(a*b)=a$ [absorption law]

Proof. ① Let B be a Boolean algebra

To prove

$a+a=a$

We know $a+0=a$

$\Rightarrow a = a+0$ $[\because a*a'=0]$

$a = a + a*a'$

$a = (a+a)*(a+a')$ (Distributive property)

$a = (a+a) * 1$ $\quad [a + a' = 1]$

$a = a + a$ $\quad [\because (a+a) * 1 = a+a$ as $a * 1 = a]$

(2) To prove

$a * a = a$

We have

$a = a * 1$ $\quad [\because a * 1 = a]$

$a = a * [a + a']$ $\quad [\because a + a' = 1]$

$a = (a * a) + (a * a')$

$a = a * a + 0.$ $\quad [a * a' = 0]$

$a = a * a$ $\quad [a + 0 = a]$

(3) To prove

$a + 1 = 1$

We know that

$1 = a + a'$ $\quad [a' = a' * 1]$

$1 = a + a' * 1$ $\quad$ [Distributivety]

$1 = (a + a') * (a+1)$ $\quad [a + a' = 1]$

$1 = 1 * (a+1)$ $\quad [1 * a = a]$

$1 = a + 1$

(4) To prove

$a * 0 = 0$

We know that $0 = a * a'$ $\quad [\because a' = a + 0]$

$0 = a * (a' + 0)$

$0 = a * a' + a * 0$ $\quad$ [Distributive law]

$0 = 0 + a * 0$ $\quad [\because a * a' = 0]$

$0 = a * 0$ $\quad [a + 0 = a]$

(5) To prove

$$a + a * b = a$$

We know that

$$a = a * 1$$

$= a * (1+b)$ [$\because 1+b=1$]

$= a*1 + a*b$ [Distributive law]

$a = a + a*b$ [$a*1 = a$]

(6) To prove

$$a * (a+b) = a$$

We know that

$$a = a + 0$$

$a = a + b*0$ [$b*0 = 0$]

$a = (a+b) * (a+0)$ [Distributive law]

$a = (a+b) * a$ [$a+0 = a$]

$a = a * (a+b)$ [commutativity]

<u>Theorem.</u> Let $\{B, +, *, ', 0, 1\}$ is a Boolean algebra. Let $a \in B$ then prove that a' is unique i.e complement of an element in Boolean algebra is unique.

Proof. Let $\{B, +, *, ', 0, 1\}$ is a Boolean algebra

let $a \in B$

then x and y be any two complements of a

then $a + x = 1, a*x = 0$ [$\because a + a' = 1$, if a' is complement of a]

and $a + y = 1, a*y = 0$ [$a*a' = 0$]

Now $x = x*1$

$x = x*(a+y)$ [$\because a+y=1$]

$x = x * a + x * y$

$x = 0 + x * y \quad [x * a = 0]$

$x = x * y \quad [a + 0 = 0]$

$x = x * y + 0$

$x = x * y + a * y \quad [\because a * y = 0]$

$x = (x + a) * y \quad$ [Distributive]

$x = 1 * y \quad [x + a = 1]$

$x = y \quad [1 * y = 1$

$\therefore$ complement of a is unique.

Theorem: In a Boolean algebra $\{B, +, *, ', 0, 1\}$

Prove $0' = 1$ and $1' = 0$

Pf Let $\{B, +, *, ', 0, 1\}$ be a Boolean algebra

let $a \in B$ then if a' is complement of a

we have $a + a' = 1$ i.e $1 + 0 = 1 \Rightarrow 0$

$$1 + 0 = 1 \Rightarrow \boxed{(1)' = 0} \quad [1 + 0 = 1]$$

Similarly $1 * 0 = 0$

$\Rightarrow 1' = 0 \quad [\because a * a' = 0]$

Theorem: Demorgan's law. Let $\{B, +, *, ', 0, 1\}$ be a Boolean algebra then for every $a, b \in B$

prove

(a) $(a+b)' = a' * b'$ (b) $(a * b)' = a' + b'$

Proof. Let $\{B, +, *, ', 0, 1\}$ be a Boolean algebra

(a) To prove
$(a+b)' = a' * b'$, $\forall a, b \in B$

Using the fact $a + a' = 1$, we have

i.e.
To prove (a) $a+b+a'*b' = 1$ and (b) $(a+b)*(a'*b') = 0$

(a)
$LHS = a+b+a'*b'$
$= b + a + a'*b'$ [commutativity]
$= b + (a+a')*(a+b')$ [Associativity]
$= b + 1*(a+b')$ [$a+a'=1$]
$= b + a + b'$ [$1*a = a$]
$= b + b' + a$
$= 1 + a$
$= 1$
$= RHS$

$\Rightarrow$

Again (b) To prove $(a+b)*(a'*b') = 0$ [$a*a' = 0$]

$LHS = (a+b)*(a'*b')$
$= [(a+b)*a']*b'$ [Associativity]
$= [a*a' + b*a']*b'$
$= [0 + b*a']*b'$
$= (a'*b)*b'$
$= a'*(b*b')$ [Associativity]
$= a'*0$
$= 0 = RHS.$
$\Rightarrow (a+b)' = a'*b'$

Similarly we can prove $(a*b)' = a' + b'$ by using duality.

i.e. (a) $a*b + a' + b' = 1$ (b) $(a*b)*(a'+b') = 0$

(a) $LHS = a*b + a' + b'$
$= b' + a' + a*b$
$= b' + (a'+a)*(a'+b)$
$= b' + 1*(a'+b)$
$= b' + a' + b$
$= b' + b + a'$ (commutativity)
$= 1 + a'$ $[b+b'=1]$
$= 1$ $[1+a'=1]$

(b) $LHS = (a*b)*[a'+b']$
$= (a*b)*a'$ (Distri)
$+ (a*b)*b'$
$= b*a*a'$
$+ a*b*b'$
$= b*0 + a*0$
$= 0+0$ $[b*0=0]$
$= 0$

Subalgebra. Let $\{B, +, *, ', 0, 1\}$ be a Boolean algebra. Let $S \subseteq B$ having 0 and 1 then $\{S, +, *, ', 0, 1\}$ is called Subalgebra if S is itself Boolean algebra.

Ex (1) Let $[B = \{\phi, (a), (b), (c), (a,b), (a,c), (b,c), (a,b,c)\}, \cup, \cap]$ be a Boolean algebra
then $S = \{\phi, (a), (b,c), (a,b,c)\}$ is Subalgebra of B.

(2) Let $\{B = \{1, 2, 5, 7, 10, 14, 15, 70\}, +, *, ', 1, 70\}$ be a Boolean algebra
then $\{S = \{1, 7, 10, 70\}, +, *, ', 1, 70\}$ is a Subalgebra.

Theorem. If S_1 and S_2 be any two subalgebras of B then $(S_1 \cap S_2)$ is also a subalgebra

Proof. Let S_1 and S_2 be any two subalgebra of a algebra B

To prove $(S_1 \cap S_2)$ is a subalgebra

let $a \in S_1 \cap S_2 \Rightarrow a \in S_1$ and $a \in S_2$

$b \in S_1 \cap S_2 \Rightarrow b \in S_1$ and $b \in S_2$

(i) (ii)

$a, b \in S_1 \Rightarrow a+b \in S_1,\ a * b \in S_1,\ a' \in S_1$ — (1)

[$\because S_1$ is a subalgebra]

$a, b \in S_2 \Rightarrow a+b \in S_2,\ a * b \in S_2,\ a' \in S_2$ — (2)

from (1) & (2), we have

$a+b \in S_1,\ a+b \in S_2 \Rightarrow (a+b) \in (S_1 \cap S_2)$

$a * b \in S_1,\ a * b \in S_2 \Rightarrow (a * b) \in (S_1 \cap S_2)$

$a' \in S_1,\ a' \in S_2 \Rightarrow a' \in (S_1 \cap S_2)$

Therefore, we have

$a, b \in (S_1 \cap S_2) \Rightarrow (a+b), (a * b) \in (S_1 \cap S_2)$ and $a' \in (S_1 \cap S_2)$

$\Rightarrow (S_1 \cap S_2)$ is algebra of B

As $(S_1 \cap S_2) \subseteq B$ and Also $0 \in (S_1 \cap S_2)$, $1 \in (S_1 \cap S_2)$

$\Rightarrow (S_1 \cap S_2)$ is subalgebra of B.

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Isomorphic Boolean Algebra

Let B and B' be any two Algebra
let $f : B \longrightarrow B'$ be one-one & onto
Then f is called an isomorphism if

$$f(a+b) = f(a) + f(b), \quad \forall a, b \in B$$
$$f(a * b) = f(a) * f(b), \quad \forall a, b \in B$$

and $f(a') = [f(a)]'$, $\forall a \in B$,

Ex

let $B = P(S)$, $S = (a, b, c)$
and $B' = \{1, 2, 5, 7, 10, 14, 35, 70\}$ be any two Algebra
then $f : B \longrightarrow B'$ is an isomorphism
satisfying $f(a+b) = f(a) * f(b)$

i.e $f(a \cup b) = f(a) \vee f(b) =$ l.c.m. of $f(a)$ & $f(b)$

$f(a \cap b) =$ g.c.d of $f(a)$ & $f(b)$
$= f(a) \wedge f(b)$

where

$f(\phi) = 1$, $f(b) = 5$, $f(a, b) = 10$
$f(a) = 2$, $f(c) = 7$, $f(a, c) = 14$
$f(b, c) = 35$, $f(a, b, c) = 70$

$f(a') = [f(a)]'$

Let $a = (a)$, $b = (a, b)$

$f[(a) \cup (a, b)] = f(a, b) = 10$ ——— (1)

$f(a) \vee f(a, b) =$ l.c.m of 2 and 5 ——— (2)
$= 10$

$\therefore$ (1) & (2) $\Rightarrow$ $f\{(a) \cup (a, b)\} = f(a) \vee f\{(a, b)\}$ etc.

Example. Prove that no Boolean algebra can have three distinct elements.

Solution. Let $\{B, +, *, ', 0, 1\}$ be a Boolean algebra.

Then B have two elements namely 0 and 1.

Let a be the third element of B. Since B is Boolean algebra it means there must exist a' in B such that

(i) $a + a' = 1$ (ii) $a * a' = 0$

Now three cases arise

(1) $a' = a$ (2) $a' = 0$ (3) $a' = 1$

[$\because$ total elements in B are now three]

Case I $a' = a$

$a + a' \Rightarrow a + a = 1 \Rightarrow a = 1$

and $a * a' = 0 \Rightarrow a * a = 0 \Rightarrow a = 0$

But a is different from 0 and 1

$\Rightarrow a' = a$ is not possible

Case II $a' = 0$ then

$a + a' = 1 \Rightarrow a + 0 = 1 \Rightarrow a = 1$, not possible

Case III. $a' = 1$ then

$a * a' = 0 \Rightarrow a * 1 = 0$

$\Rightarrow a = 0$, not possible

Hence, $\{B, +, *, ', 0, 1\}$ has only two distinct elements

i.e no Boolean algebra can have three distinct elements.

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Question. Prove that for any $a, b, c \in B$ following are equal

(a) $(a+b)(a'+c)(b+c)$ (b) $ac + a'b + bc$

(c) $(a+b)(a'+c)$ (d) $ac + a'b$

Solution. Let $\{B, +, *, ', 0, 1\}$ be a Boolean algebra

(1) To show (a) = (b).

$(a)\ (a+b)(a'+c)(b+c)$

$(a) = (a+b)(c+a')(c+b)$. (commutativity)

$(a) = (a+b)(c+a'b)$ [Distributivity]

$(a) = a(c+a'b) + b(c+a'b)$ (Distributivity)

$(a) = ac + aa'b + bc + ba'b$

$(a) = ac + 0 + bc + a'b$ [$bb = b$]

$(a) = ac + a'b + bc$

$(a) = (b)$

$\Rightarrow (a) = (b)$

(2) To show (c) = (b)

$(c) = (a+b)(a'+c)$

$(c) = aa' + ac + ba' + bc$

$(c) = 0 + ac + a'b + bc$

$(c) = (b)$

(3) $(b) = (d)$

$(b) = ac + a'b + bc$

$(b) = ac + a'b + (a+a')bc$

$(b) = ac + a'b + abc + a'bc$

$(b) = ac(1+b) + a'b(1+c) = ac + a'b$ $\left[\begin{array}{l} 1+b=1 \\ 1+c=1 \end{array}\right]$

from (1), (2), (3)

(a) = (b) = (c) = (d). proved.

Question. In any Boolean algebra B, prove

(1) $(a+b)(b+c)(c+a) = ab+bc+ca, \forall a,b,c \in B$

(2) $(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$

Solution. (1) $LHS = (a+b)(b+c)(c+a)$

$= (a+b)[(b+c)(c+a)]$ [Associativity]

$= (a+b)[bc+ba+c+ca]$ [$c.c=c$]

$= (a+b)[bc+ba+c]$ [$c+ca=c$ Absorption law]

$= (a+b)[ba+c+bc]$

$= (a+b)[ba+c]$ [$c+bc=c$ Absorption law]

$= a(ba+c)+b(ba+c)$ [$bb=b$ etc]

$= ab+ac+ba+bc$

$= ab+bc+ca$

$LHS = RHS$

i.e $\boxed{(a+b)(b+c)(c+a) = ab+bc+ca}$

(2) Let B be a Boolean algebra.

To Prove $(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$

$(a+b')(b+c')(c+a')$

$= [ab+ac'+b'b+b'c'][c+a']$

$= [ab+ac'+b'c'][c+a']$

$= abc+ac'c+b'c'c+aba'+ac'a'+a'b'c'$

$= abc+0+0+ \quad +0+a'b'c'$

$= abc+a'b'c'$ —— (1)

$(a'+b)(b'+c)(c'+a)$

$= (a'+b)[(b'+c)(c'+a)]$

$= (a'+b)[b'c' + b'a + cc' + ca]$

$= (a'+b)[b'c' + b'a + ca]$ $\quad [cc'=0]$

$= a'(b'c' + b'a + ca) + b(b'c' + b'a + ca)$

$= a'b'c' + 0 + 0 + 0 + 0 + bca$

$= abc + a'b'c'$ ——— (2)

(1) & (2) $\Rightarrow$

$\boxed{(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)}$

Example. In Boolean algebra B

if $a+x = b+x$

and $a+x' = b+x'$

$\therefore$ prove $a=b$

Solⁿ. In Boolian algebra B

given $a+x = b+x$ ——— (1)

$a+x' = b+x'$ ——— (2)

To prove $a=b$

$a = a+0$ $\quad [a+0=a]$

$a = a+xx'$

$a = (a+x)(a+x')$

$a = (b+x)(b+x')$

$a = b+xx'$

$a = b+0$

$a = b$, proved.

Remember: In Boolean algebra $a \le b$ if $ab'=0$

$a \le b \Leftrightarrow a+b=b$ &

$\boxed{\text{i.e. } ab'=0 \Leftrightarrow a \le b}$

Boolean function

Let $\{B, +, *, ', 0, 1\}$ be a Boolean algebra.

Let $a, b \in B$ then

Expression such as

$ab,\ a'b,\ ab',\ (a+b')',\ a'+b'$ etc

are called Boolean functions.

min term. A Boolean expression of n variables namely $x_1, x_2, ---, x_n$ is said to be min term or a minimal polynomial if it is of the form

$$f_1(x_1), f_2(x_2), ----, f_n(x_n)$$

where $f_i(x_i) = x_i$ or x_i' $\forall\ i = 1, 2, ---, n$.

ex $x_1x_2,\ x_1x_2',\ x_1'x_2,\ x_1'x_2'$ are example of min terms

OR

$\cdot$	x_2	x_2'
x_1	x_1x_2	x_1x_2'
x_1'	$x_1'x_2$	x_2'

In C.T. we see that entries in C.T are called min terms namely x_1x_2, x_1x_2', $x_1'x_2$, x_2' these are 2^n in terms here $n=2$ (i.e x_1 & x_2)

If x_1, x_2, x_3 are there then no of min terms can be calculated as

	$x_1 x_2$	$x_1 x_2'$	$x_1' x_2$	$x_1' x_2'$
x_3	$x_1 x_2 x_3$	$x_1 x_2' x_3$	$x_1' x_2 x_3$	$x_1' x_2' x_3$
x_3'	$x_1 x_2 x_3'$	$x_1 x_2' x_3'$	$x_1' x_2 x_3'$	$x_1' x_2' x_3'$

Total no of minterms are 2^n $(n=3, x_1, x_2, x_3)$

$= 2^3 = 8$

namely $x_1x_2x_3, x_1x_2'x_3, x_1'x_2x_3, x_1'x_2'x_3$

$, x_1x_2x_3', x_1x_2'x_3', x_1'x_2x_3', x_1'x_2'x_3'$

Total of 8

Similarly we can go further for $n=4$ i.e x_1, x_2, x_3, x_4 etc.

Note Here $x_1x_2x_3$ may also be written as

$f_1(x_1)\, f_2(x_2)\, f_3(x_3)$

i.e. $x_1x_2x_3 = f_1(x_1) f_2(x_2) f_3(x_3)$
$x_1'x_2x_3' = f_1(x_1) f_2(x_2) f_3(x_3)$ $\Big]$ as $f_i(x_i) = x_i$ or x_i'.

Disjunctive normal form

A Boolean function in n variables $x_1, x_2, \ldots, x_n$ is said to be in disjunctive normal form if the function is a sum of terms of the type

$f_1(x_1) f_2(x_2) f_3(x_3) \cdots f_n(x_n)$

where $f_i(x_i) = x_i$ or x_i'.

i.e DN forms we mean sum of min terms

Question. (1) Write the function

$f(x,y,z) = (xy' + xz)' + x'$ in DN form.

$f(x,y,z) = (xy')' \cdot (xz)' + x'$, by Demorgan's law

$f(x,y,z) = (x'+y) \cdot (x'+z') + x'$

$f(x,y,z) = x' + yz' + x'$ Distributive law

$f(x,y,z) = x' + yz'$ $[x'+x' = x']$

$f(x,y,z) = x'(y+y')(z+z') + (x+x')\, yz'$

$[\because x+x'=1 \text{ etc}]$

$f(x,y,z) = x'(yz + yz' + y'z + y'z') + xyz' + x'yz'$

$f(x,y,z) = x'yz + x'yz' + x'y'z + x'y'z' + xyz' + x'yz'$

$f(x,y,z) = x'yz + x'yz' + x'y'z + x'y'z' + xyz'$

$[a+a=a]$

which is in required D.N. form.

Note: If we are asked to write f' (complement of f) then we have to find all $2^3 = 8$ min terms as

(1) x, y

(2)

	xy	$x'y$	xy'	$x'y'$
z	xyz	$x'yz$	$xy'z$	$x'y'z$
z'	xyz'	$x'yz'$	$xy'z'$	$x'y'z'$

Now, we (✓) the present of min. terms of Q (1)

Now we write rest elements as ~~su~~ their sum $(xyz + xy'z + xy'z')$ which is called Complement of f in Q(1) as just above

i.e $f' = xyz + xy'z + xy'z'$

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Question. 02. Write the Boolean function $f = x_1 + x_2$ in sum of products canonical form in three variables x_1, x_2, x_3. (i.e. in DN form)

Solution. Let

$$f(x_1, x_2, x_3) = x_1 + x_2$$

$$= x_1(x_2 + x_2')(x_3 + x_3') + x_2(x_1 + x_1')(x_3 + x_3')$$

$$f(x_1, x_2, x_3) = x_1(x_2x_3 + x_2x_3' + x_2'x_3 + x_2'x_3')$$
$$+ x_2(x_1x_3 + x_1x_3' + x_1'x_3 + x_1'x_3')$$

$$f(x_1, x_2, x_3) = x_1x_2x_3 + x_1x_2x_3' + x_1x_2'x_3 + x_1x_2'x_3'$$
$$+ x_1x_2x_3 + x_1x_2x_3' + x_1'x_2x_3 + x_1'x_2x_3'$$

Note: Here complement of f is zero
i.e $f' = 0$.

Question. Find and simplify the function by the table as

Row	x	y	z	$f(x,y,z)$
1	1	1	1	0
2	1	1	0	1 ✓
3	1	0	1	0
4	1	0	0	1 ✓
5	0	1	1	0
6	0	1	0	0
7	0	0	1	0
8	0	0	0	0

Soln. From the given table $f(x,y,z)$ can be written as

$$f(x,y,z) = xyz' + xy'z'$$

For simplification

$$f(x,y,z) = xz'(y + y') = xz' \quad [\because y + y' = 1]$$

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Question. Express

$$f(x,y,z) = (x+y)(x+y')(x'+z)$$

in DN form in three variables

Soln Given

$$f(x,y,z) = (x+y)(x+y')(x'+z)$$

$$f(x,y,z) = (x+yy')(x'+z)$$

$$f(x,y,z) = x(x'+z) \qquad [\because yy'=0]$$

$$f(x,y,z) = xx'+xz$$

$$f(x,y,z) = xz \qquad [\because xx'=0]$$

$$f(x,y,z) = x(y+y')z$$

$$f(x,y,z) = xzy+xzy'$$

$$f(x,y,z) = xyz+xy'z$$

Aliter Its table is given by

x	y	z	$f(x,y,z)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

Question. In a Boolean algebra, show
imp
$$f(x,y) = x f(1,y) + x' f(0,y)$$

Solution We know that for any two variables DN form of $f(x,y)$ is given by

$$f(x,y) = xy + xy' + x'y + x'y' \quad \text{--- (1)}$$

put $x=1 \Rightarrow x'=0$

i.e $f(1,y) = y+y'$ ——— (2)

put $x=0$ in (1) then $x'=1$

$$f(0,y) = y+y' \quad \text{——— (3)}$$

Now, eqn (1) can be re-written as

$$f(x,y) = x(y+y') + x'(y+y')$$

$$\boxed{f(x,y) = x f(1,y) + x' f(0,y)} \quad [\text{from (2) \& (3)}]$$

, proved

Question. Express each of the following in DN form in smallest possible no of variables.

Soln (i) $xy' + xz + xy$

(ii) $(x'y + xyz' + xy'z + x'y'z't + t')'$

Soln (i) let $f = xy' + xz + xy$

$$f = f(x,y) = x(y+y') + xz$$

$$= x + xz \quad [y+y'=1]$$

$$= x(1+z)$$

$$f(x,y) = x \quad [1+z=1]$$

(ii) Let

$$f'(x,y,z,t) = (x'y + xyz' + xy'z + x'y'z't + t')'$$

then

$$f(x,y,z,t) = x'y + xyz' + xy'z + x'y'z't + t'$$

$$f(x,y,z,t) = x'y(z+z')(t+t') + xyz'(t+t') + xy'z(t+t') + x'y'z't + (x+x')(y+y')(z+z')t'$$

$$f = x'y(zt+zt'+z't+z't') + xyz't + xyz't' + xy'zt + xy'zt' + x'y'z't + (xt'+x't')(yz+yz'+y'z+y'z')$$

$$f = x'yzt + x'yzt' + x'yz't + x'yz't' + xyz't + xyz't' + xy'zt + xy'zt' + x'y'z't + xyzt' + xyz't' + xy'zt' + xy'z't' + x'yzt' + x'yz't' + x'y'zt' + x'y'z't'$$

$$f = x'yzt\,(1) + x'yzt'\,(2) + x'yz't\,(3) + x'yz't'\,(4) + xyz't'\,(5) + xy'zt\,(6) + xy'zt'\,(7) + x'y'z't\,(8) + xyzt'\,(9) + xy'z't'\,(10) + x'y'zt'\,(11) + x'y'z't'\,(12) + xyz't\,(13)$$

(total 13 terms)

	xy	$x'y$	xy'	$x'y'$
z	xyz	$x'yz$	$xy'z$	$x'y'z$
z'	xyz'	$x'yz'$	$xy'z'$	$x'y'z'$

	xyz	$x'yz$	$xy'z$	$x'y'z$	xyz'	$x'yz'$	$xy'z'$	$x'y'z'$
t	$xyzt$	$x'yzt$ (1)	$xy'zt$ (6)	$x'y'zt$	$xyz't$ (13)	$x'yz't$ (3)	$xy'z't$	$x'y'z't$ (8)
t'	$xyzt'$ (9)	$x'yzt'$ (2)	$xy'zt'$ (7)	$x'y'zt'$ (11)	$xyz't'$ (5)	$x'yz't'$ (4)	$xy'z't'$ (10)	$x'y'z't'$ (12)

$$\therefore f'(x,y,z) = xyzt + x'y'zt + xy'z't \quad \text{Ans}$$

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Max. terms In Boolean algebra, the expression of the type

$[f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)]$ is called max-terms where $f_i(x_i) = x_i$ or x_i'.

Conjuctive normal form A function f of n variables namely $x_1, x_2, \ldots x_n$ is called in CN form if it is written as a product of max. terms

$$f = \{f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)\}\{f_1(x_1) + f_2(x_2) + \cdots f_n(x_n)\}$$

$$+ \cdots\cdots .$$

Example Let x_1, x_2 be two variable then

$f_1(x_1) = x_1$ or $f_1(x_1) = x_1'$

$f_2(x_2) = x_2$ or $f_2(x_2) = x_2'$

$\therefore f(x_1, x_2) = (x_1 + x_2)(x_1 + x_1')(x_2 + x_2')$ etc

If $f(x, y) = xy$ then we can write it in CN form as

$$f(x, y) = (x + 0)(y + 0)$$

$$= (x + yy')(y + xx') \quad [\because yy' = 0, xx' = 0]$$

$$f(x, y) = (x + y)(x + y')(y + x)(y + x')$$

Example. Write the function,

$(xy' + xz)' + x'$ in CN form.

Solution. Let $f(x,y,z) = (xy'+xz)' + x'$

$= (xy')'(xz)' + x'$

$= (x'+y)(x'+z') + x'$

$= (x' + yz') + x'$

$= x' + yz'$

$= (x'+y)(x'+z')$

$= (x'+y+zz')(x'+z'+yy')$

$= (x'+y+z)(x'+y+z')(x'+z'+y) \cdot (x'+z'+y')$

$f(x,y,z) = (x'+y+z)(x'+y+z')(x'+y'+z')$

which is in required CN form. Ans [as second and 3rd terms are same]

Question. For any Boolean fn, show

$f(x_1,x_2) = [x_1 + f(0,x_2)][x_1' + f(1,x_2)]$

Soln let In a Boolean function

$f(x_1,x_2) = (x_1+x_2)(x_1+x_2')(x_1'+x_2)(x_1'+x_2')$ — (1)

$= (x_1 + x_2x_2')(x_1' + x_2x_2')$

put $x_1 = 1$ then $x_1' = 0$ into (1)

$f(1,x_2) = (1 + x_2x_2')(0 + x_2x_2')$

$f(1,x_2) = x_2x_2'$ —— (2) $[1+a = a]$

put $x_1 = 0$, $x_1' = 1$

(1) $\Rightarrow$

$$f(0, x_2) = (0 + x_2 x_2')(1 + x_2 x_2')$$
$$= (x_2 x_2')(1) \quad [\because 1 + a = a]$$
$$f(0, x_2) = x_2 x_2' \quad \text{———} \quad (3)$$

$\because$ (1), (2) & (3) $\Rightarrow$

$$f(x_1, x_2) = [x_1 + f(0, x_2)][x_2 + f(1, x_2)]$$

from (3) , from (2)

, proved.

<u>Note</u> In DN form if $x = 1$ then in CN form $x = 0$

<u>Example.</u> Find Boolean function f in CN form that is given by .

x	y	z	$f(x,y,z)$	
1	1	1	1	
1	1	0	0	✓
1	0	1	0	✓
1	0	0	0	✓
0	1	1	1	
0	1	0	1	
0	0	1	1	
0	0	0	0	✓

In CN

$x' = 1$
$y' = 1$
$z' = 1$

$x = 0$
$y = 0$
$z = 0$

<u>Sol</u> From the table, we have (in CN form)

$$f(x,y,z) = (x' + y' + z)(x' + y + z')(x' + y + z) \times (x + y + z)$$

Minimization of Boolean functions :

Karnaugh map (or K-map)

The K-map is a pictorial representation of truth table of the Boolean function.

Let $f = xy + x'y$ —(1)

We know that:

	x	x'
y	xy	$x'y$
y'	xy'	$x'y'$

So K-map of (1) is given by

	x	x'
y	1	1
y'	0	0

$f = xy + x'y$ $[\because x + x' = 1]$

$f = y$

Example. Use the K-map method to find a minimal ON form of the following functions

(a) $f(x, y) = xy + xy'$

(b) $f(x, y) = xy + x'y + x'y'$

(c) $f(x, y) = xy + x'y'$

(a) K-map for $f(x, y) = xy + xy'$

	x	x'
y	1	0
y'	1	0

$\Rightarrow f(x, y) = x$
(minimal form)

(b) K- map for $f(x,y) = xy + x'y + x'y'$

	x	x'
y	1	1
y'	0	1

$\Rightarrow f(x,y) = y + x'$
(Minimal form)

(c) K- map for $f(x,y) = xy + x'y'$:

	x	x'
y	1	0
y'	0	1

$\Rightarrow f(x,y) = xy + x'y'$
(Minimal form)

Example. Use K- map method to find a minimal DN form of the following functions

(a) $f(x,y) = xyz + xyz' + x'y'z + x'yz'$

(b) $f(x,y) = xyz + xy'z + xyz' + x'y'z + x'yz$

(c) $f(x,y) = xyz + xyz' + x'y'z + x'y'z' + x'yz'$

(a) K- map of given function

	xy	xy'	$x'y$	$x'y'$
z	1	0	1	0
z'	1	0	0	1

Req. minimal function $= xy + x'yz + x'y'z'$

(b) K- map of given function

	xy	xy'	$x'y$	$x'y'$
z	1	1	1	1
z'	1	0	0	0

Req. minimal fn $= z + xy$

K-map for four variables

	xy	$x'y$	xy'	$x'y'$
zw				
$z'w$				
zw'				
$z'w'$				

Example. Use K-map technique to find minimal form for the Boolean function.

Solution. (a) $f = x'yzw + xy'zw' + x'y'zw' + xyz'w' + xy'z'w'$

(b) $f = xy' + xyz + x'y'z' + x'yzw'$

(a)

	xy	xy'	$x'y'$	$x'y$
zw	0	0	0	1
zw'	0	1	1	0
$z'w'$	1	1		
$z'w$	0	0		0

Req min fun

$$f = x'yzw + y'zw' + xz'w'$$

(b)

	xy	xy'	x'y'	x'y
zw	1	1		
zw'	1	1		1
z'w'		1	1	
z'w		1	1	

Req

$$f = xz + y'z' + \underline{x'yzw' + xyzw'}$$

$$= xz + y'z' + yzw'(x' + x)$$

$$f = xz + y'z' + yzw'$$

Ans.